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Invariants, and Equations Associated with the General Linear Differential Equation

Introduction

The formation of functions associated with differential equations and analogous to the invariants of algebraic equations has occupied the attention of several Mathematicians for some years because of their great value in obtaining practical as well as theoretical solutions.

Starting with the work of M. Laguerre and of Professor Brioschi; M. Halphen in two important memoirs* indicated a method for the formation of invariants, but involving very difficult analysis. He derived the two simplest invariants for the Cubic and Quartic, and such derivatives as may be deduced from them.

* See, p. III

[illegible] by means of the trans-formation $y = u e^{-\int \frac{R_1}{R_0} dx}$ brings the equation to a form having zero for the coefficient of the second term.

[illegible] published a [illegible] valuable memoir in which by means of the following transformation he obtains a canon-ical form in which the coefficients of both the second and third terms vanish. It is thus stated;

When the linear differential equation

$$\frac{d^n y}{dx^n} + \frac{n(n-1)}{1\cdot 2} P_2 \frac{d^{n-2}y}{dx^{n-2}} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} P_3 \frac{d^{n-3}y}{dx^{n-3}} + \dots + P_n y = 0$$

has its dependent variable y transformed to u by the equation $y = u\lambda$, λ being a function of x, and its independent variable changed from x to z, where z and λ are determined [illegible]

$$\lambda = \theta^{n-1}, \quad \frac{dz}{dx} = \theta^{-2}, \quad \frac{d^2\theta}{dx^2} + \frac{3}{n+1} P_2 \theta = 0 \qquad (1), \ (2)$$

the transformed [illegible] is in the canonical form,

$$\frac{d^n u}{dz^n} + \binom{n}{3} \alpha_3 \frac{d^{n-3}u}{dz^{n-3}} + \binom{n}{4} \alpha_4 \frac{d^{n-4}u}{dz^{n-4}} + \dots + \alpha_n u = 0$$

The coefficients [illegible]

† see pp IV

tions are so connected that there exists $n-2$ algebraically independent functions $\theta_\sigma(x)$ of the coefficients P and their derivatives, which are such that, when the same function $\theta_\sigma(z)$ is formed of the coefficients Q and their derivatives, the equation

$$\theta_\sigma(x) = \xi'^{\sigma}\,\theta_\sigma(z) \qquad (3)$$

is identically satisfied.

For this form of the differential equation,

$$\theta_\sigma(z) \equiv Q_\sigma + \frac{\sigma}{2}\sum_{r=1}^{r=\sigma-3}(-1)^r a_{r,\sigma}\,\frac{d^r Q_{\sigma-r}}{dz^r}. \quad \text{where,} \qquad (4)$$

$$a_{r,\sigma} = \frac{\sigma-1!\,\sigma-2!\,2\sigma-r-2!}{r!\,2\sigma-3!\,\sigma-r!\,\sigma-r-1!}\ *$$

Thus $\theta_\sigma(z)$ is independent of the order of the equation. In this z is completely determin-

* "Mémoire sur la réduction des équations differentielles linéaires aux formes intégrables" 'Mémoires des Savants Étrangers,' Vol. 28 No. 1, 301 pp, 1880 Also "Sur les invariants des équations differentielles linéaires du quatrième ordre," 'Acta Math.' vol. 3 1883 pp. 325–380.

ed by equations (1) and (2); but there may be difficulties in the way of solving (2) and thus it is desirable to form the invariants for the uncanonical form of the equation.

For this purpose Mr. Forsythe establishes relations between the coefficients P and Q for the case in which z, being arbitrary, is given the value $x + \epsilon\mu$ where ϵ is so small that the square and higher powers may be neglected, and μ is an arbitrary non-constant function of x. These relations are expressed thus:—

$$Q = [illegible] - \frac{1}{2}\epsilon \sum_{\theta=0}^{\theta=s-1}\left[\frac{s!}{\theta!\,(s-\theta+1)!}\left\{n(s-\theta-1)+s+\theta-1\right\}P_\theta \frac{d^{s-\theta+1}\mu}{dx^{s-\theta+1}}\right] \quad (5)$$

These relations are fully developed in Mr. Forsythe's memoir and also in Dr. Craig's Treatise on Linear Differential Equations Vol. I Chap. XIII.

† "Invariants, Covariants, and Quotient Derivatives Associated with Linear Differential Equations" Philosophical Transactions of the Royal Society of London Vol. 179 (1888) A. pp 377–489.

$$\frac{d^r Q_s}{dx^r} = \frac{d^r P_s}{dx^r}\left\{1 - (r+1)\epsilon \text{[illegible]}\right\} \text{[illegible]} P_s \frac{d^{r+1}\mu}{dx^{r+1}}$$

$$- \epsilon \sum_{m=1}^{m=r-1} \left[\frac{r!}{m!\,(r-m+1)!}\left\{s(r+1)-m(s-1)\right\} \frac{d^m P_s}{dx^m}\,\frac{d^{r-m+1}\mu}{dx^{r-m+1}}\right] \quad (6)$$

$$- \frac{\epsilon}{2} \sum_{t=0}^{t=\text{[illegible]}} \left[\text{[illegible]}\right] \frac{d^r}{dx^r}\left(P_t \frac{d^{s-t+1}\mu}{dx^{s-t+1}}\right)$$

[illegible] then calculate the coefficients of θ_3, θ_4, θ_5, θ_6 and θ_7 for the equation in the [illegible] form.

In section I. following this introduction, the general invariant θ_s is considered and it is there shown that, in the non-linear part every term is of the form $A.B.C.$ where A is a number, B is a function of P_2 and its derivatives, and C is an invariant or the derivative of an invariant with suffix differing from s by an even number. When s is even C may be a number.

Section II deals with the coefficients of θ_s, giving some general expressions [illegible]

given value of s.

Section III. treats of Associate variables and Associate Equations, showing which are identical and which may not be.

On pp. 293–295 of his Treatise Dr. Craig has shown that, for the Quartic, Sextic and Octic, the condition that they be self-adjoint [illegible] the invariants with odd suffix vanish, and also announces a theorem which is proven in Section IV. The proof as given at that time only applied to equations in Mr. Forsyth's Canonical form. By aid of what is established in section I. it is shown to apply also to equations [illegible] form.

A fuller presentation of the subject will be found in the works to which reference has been made.

It was at the suggestion of Dr Craig that sections III and IV were begun. The whole paper has [illegible] valuable criticism.

The Form of the General Linear Primeinvariant θ_s

Section I

Since θ_s has only a linear part when P_2 vanishes its form must be as follows:—

$$\Big[A P_s + B P^{(1)}_{s-1} + C P^{(2)}_{s-2} + D P^{(3)}_{s-3} + E P^{(4)}_{s-4} + \ldots M P_2^{(s-2)}\Big]$$
$$+\Big[P_2\{a_{22}\,\theta_{s-2} + a_{33}\,\theta^{(1)}_{s-3} + a_{44}\,\theta^{(2)}_{s-4} + a_{55}\,\theta^{(3)}_{s-5} + \ldots\ a_{s-2,s-2}\,P_2^{(s-4)}\}\Big]$$
$$+\Big[P_2^{(1)}\{\qquad b_{33}\,\theta_{s-3} + b_{44}\,\theta^{(1)}_{s-4} + b_{55}\,\theta^{(2)}_{s-5} + \ldots\ b_{s-2,s-2}\,P_2^{(s-5)}\}\Big]$$
$$+\Big[P_2^{(2)}\{\qquad c_{44}\,\theta_{s-4} + c_{55}\,\theta^{(1)}_{s-5} + \ldots\ c_{s-2,s-2}\,P_2^{(s-6)}\}\Big]$$
$$+\Big[P_2^2\{\qquad d_{44}\,\theta_{s-4} + d_{55}\,\theta^{(1)}_{s-5} + \ldots\ d_{s-2,s-2}\,P_2^{(s-6)}\}\Big]$$
$$+\Big[\qquad\qquad\qquad\Big]$$
$$+\Big[\qquad\qquad\qquad\Big]$$

etc. etc.

In this (r) is the differential index, so that $P_2^{(r)} \equiv \dfrac{d^r P_2}{dx^r}$ $\quad \theta^{(r)}_{s-k} \equiv \dfrac{d^r \theta_{s-k}}{dx^r}$. The sum of the suffixes and differential indices, it will be noticed, equals s for every term, that is, it possesses a kind of homogeneity.‡ s is called the index or dimension number of θ_s: the dimension number of $P_2^2\,\theta^{(w)}_{s-k}$ being

‡ Philosophical Transactions, [illegible]

$(p+2)r+u+s-k$. Denoting the terms within the square parenthesis by $L, \alpha, \beta, \gamma, \delta$, etc. $P_s \equiv L+\alpha+\beta+\gamma+\delta+$ etc.

The notation used here will [illegible] that multiplication [illegible] that is [illegible] the "μ" and all its derivatives, of the equation.

$z = x + \varepsilon\mu$ will be dropped, that is, they will be treated as unity, when the result will not be changed by doing so. Also

$P_1^{(\tau)} \equiv \frac{d^{(\tau)} P_2}{dx^{\tau}}$ will be considered [illegible] P_2

Mr. Forsyth has given the general form of the term in L viz. $(-1)^{T}\frac{s(s-2)(2s-T-2)}{2(T)(T+?)(s-T)(s-3)}P_2^{(T)}$ [illegible]

I now desire to show that, when k is odd, each of the numerical coefficients a_{KK}, b_{KK}, c_{KK}, d_{KK} etc. of the non-linear part of θ_s [illegible] zero.

From page III of the introduction we have $P_2(x) = z^{(?)} P_2(z) \cdot (1+\varepsilon\mu')^{(?)}$ identically [illegible] in the right member of this identity the P_2 and their derivatives [illegible]

tives are replaced by their values in terms of the P's and their derivatives, as expressed by formulæ (M: IV Intr.)

$$Q_s = P_s \{1 - [illegible] \epsilon \mu'\} - \tfrac{1}{2}\epsilon \sum_{\theta=[illegible]}^{\theta=s-1} \left[\frac{s!}{[illegible]\,\theta+1!}\right] \{n(s-\theta-1)+s+\theta-1\} P_\theta \, \mu^{(s-\theta+1)} \qquad (5)$$

$$\frac{d^2 Q_s}{dz^2} = P_s^{(2)} \{1-(2+s)\epsilon\mu'\} - s\epsilon P_s \mu^{(2+1)}$$
$$- \epsilon \sum_{m=1}^{m=2-1} \left[\frac{2!}{m!\,2-m+1!}\right] \{s(2+1) - m(s-1)\} P_s^{(m)} \mu^{(2-m+1)}$$
$$- \tfrac{1}{2}\epsilon \sum_{\theta=\epsilon}^{\theta=s-1} \left[\frac{s!}{\theta!\,s-\theta+1!}\right] \{n(s-\theta-1)+s+\theta-1\} \left(P_\theta \mu^{(s-\theta+1)}\right)^{(2)} \qquad (6)$$

or, remembering the convention $P_2^{(0)} = P_2$, as for both (a) ... (1).

express by the formula;

$$\frac{d^2 Q_s}{dz^2} = P_s^{(2)} \{1-(2+s)\epsilon\mu'\}$$
$$- \epsilon \sum_{m=0}^{m=2-1} \left[\frac{2!}{m!\,2-m+1!}\right] \{s(2-m+1)+m\} P_s^{(m)} \mu^{(2-m+1)} \qquad ([illegible])$$
$$- \tfrac{1}{2}\epsilon \sum_{\theta=[illegible]}^{\theta=s-1} \left[\frac{s!}{\theta!\,[illegible]+1!}\right] \{n(s-\theta-1)+s+\theta-1\} \left(P_\theta \mu^{(s-\theta+1)}\right)^{(2)}$$

[illegible] giving values $= 0, 1, 2, 3 \ldots\ldots s$, then the terms with dimension "s" in each member cancel, those of dimension "$s-1$" furnish the numerical coefficients

in L. and there remain terms of dimension $s-2$ and less with which to determine the coefficients in the non-linear part.

To the preceding formulae may be added

$$\frac{d^2 P_s(x)}{dx^2} = \frac{d^2 [illegible] P_s(z)}{dz^2} z'^2 = ([illegible]) \frac{d^2 P_s(z)}{dz^2} \quad (8)$$

$$\frac{d^2 P_s(z)}{dz^2} = P_s^{(2)}(z) \{ [illegible] - [illegible] \sum_{m=2} [illegible] \} \quad (9)$$

Then. $Q_2 = P_2(1-2\varepsilon\mu') - \frac{n+1}{6}\varepsilon\mu^{(3)}$

$Q_3 = P_3(1-3\varepsilon\mu') - 3\varepsilon\mu'' P_2 - \frac{n+1}{4}\varepsilon\mu^{(4)}$

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$$Q_6 = P_6(1-6\varepsilon\mu') - 15\varepsilon\mu'' P_5 - \tfrac{5}{2}(n+7)\varepsilon\mu''' P_4 - 5(n+4)\varepsilon\mu^{(4)} P_3 - \tfrac{3}{2}(3n+7)\varepsilon\mu^{(5)} P_2 - \frac{5(n+1)}{14}\varepsilon\mu^{(7)}$$

$\frac{dQ_2}{dz} = P_2^{(1)}([illegible]) - [illegible] - \frac{n+1}{6} [illegible]$

$\frac{d^2 Q_2}{dz^2} = P_2^{(2)}(1-4\varepsilon\mu') - 5\varepsilon\mu'' P_2^{(1)} - 2\varepsilon\mu^{(3)} P_2 - \frac{n+1}{6}\varepsilon\mu^{v}$

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$$\frac{d^2 Q_4}{dz^2} = P_4^{(2)}(1-6\varepsilon\mu') - 9\varepsilon\mu'' P_4' - [illegible]\varepsilon\mu''' P_? - [illegible] \{ [illegible] \} - \frac{3}{16}(n+1)\varepsilon\mu^{(8)}$$

From which follow

$Q_2^{r} = P_2^{r}(1-2r\varepsilon\mu') - r P_2^{r-1}\frac{n+1}{6}\varepsilon\mu^{(3)}$.

$\left(\frac{d^2 Q_2}{dz^2}\right)^m = P_2^{(2)m}(1-m4\varepsilon\mu') - m P_2^{(2)m-1}\{5\varepsilon\mu'' P_2^{(1)} + 2\varepsilon\mu''' P_2 + \frac{n+1}{6}\varepsilon\mu^{v}\}$.

If $P_2''' \theta_{s-K}^{(K-5)(1)}$ be a term in θ_s then will the term $\frac{d^3 Q_2}{dz^3} \frac{d^{K-5}\theta_{s-K}^{(2)}}{dz^{K-5}}$ be multiplied by [illegible] $(1+5\varepsilon\mu')$ [illegible]

$$P_2''' \theta_{s-K}^{(K-5)(1)} = (1+5\varepsilon\mu')\frac{d^3Q_2}{dz^3}\frac{d^{K-5}\theta_{s-K}^{(2)}}{dz^{K-5}}$$

$$= \{1+5\varepsilon\mu'\}\{P_2'''(1 - [illegible]) - 9\varepsilon\mu''P_2'' - 7\varepsilon\mu'''P_2' - [illegible] - \tfrac{n+1}{6}\varepsilon\mu^{iv}\} \times$$

$$\left(\theta_{s-K}^{(K-5)}\{1-(5-?)\varepsilon\mu'\} - \varepsilon \sum_{m=0}^{m=K-6} \frac{K-5!}{m!\,K-4-m!}\{[illegible]\}\theta_{s-K}^{(m)}\mu^{(K-4-m)}\right)$$

$$= P_2'''\theta_{s-K}^{(K-5)} - \theta_{s-K}^{(K-5)}\left(9\varepsilon\mu''P_2'' + 7\varepsilon\mu'''P_2' + 2\varepsilon\mu^{iv}P_2 + \tfrac{n+1}{6}\varepsilon\mu^{v}\right)$$

$$- P_2'''\varepsilon \sum_{m=0}^{m=K-6} \frac{K-5!}{m!\,K-4-m!}\{(K-4-m)(s-K)+m\}\theta_{s-K}^{(m)}\mu^{(K-4-m)}$$

In this equation the terms of dimension s cancel and $-\varepsilon$ is a factor of the remaining terms, so that when every term of θ_s is treated in this way, all terms of dimension s cancel each other and the remainder is divisible by $-\varepsilon$. Denoting by $R.L.$ the remainder of the linear part L, by R_α the remainder of terms in α etc. and by $\binom{n}{2}$ the binomial coefficient $\frac{n!}{2!\,n-2!}$ and also omitting the μs, $-\varepsilon$ being divided out $R.L.$ gives

$$[illegible]\; s-1\, P_{s-1} + \tfrac{s.s-1}{2.1}(n+1+2(s-2))P_{s-2} + [illegible] \left\{\binom{s}{K}\tfrac{n+1}{2}\tfrac{K-1}{K+1} + \binom{s}{K+1}\right\}P_{s-K} \quad K = 1, 2, 3, 4 \ldots s$$

$$+ B_1\; s-1\, P_{s-1} + [illegible] \text{ or } \sum_{K=1}^{K=s}\left[\left\{\binom{s-1}{K-1}\frac{n+1}{2}\frac{K-2}{K} + \binom{s-1}{K}\right\}\left(\mu^{(K+1)}P_{s-K}\right)^{(1)}\right]$$

$$+C\Big[s-2\,P_{s-2}+ \;+\frac{2!}{1\cdot 2!}\{2(s-2)+1\}P'_{s-2}+\sum_{K=3}^{K=s}\left\{\binom{s-2}{K-2}\frac{n+1}{2}\frac{K-3}{K-1}+\binom{s-2}{K-1}\right\}\left(P_{s-K}\mu^{(K+1)}\right)''$$

for the first three terms of θ_s. Or putting for [illegible] etc. their values (ω,μ,ρ2) · 1, [illegible]

etc. the $(r+1)$st. term gives

$$(-1)^r\frac{s!\ [illegible]}{[illegible]\,(s-2)!\,(s-r-1)!\,(2s-[illegible])}\left[\left\{\binom{r}{m}(s-r)+\binom{r}{m-1}\right\}P^{(m)}_{s-r}\quad m=0,1,2\ \cdots\ r-1\right.$$
$$\left.+\left\{\binom{s-2}{K-2}\frac{n+1}{2}\frac{K-r-1}{K-r+1}+\binom{s-r}{K-r+1}\right\}\left(\mu^{K+1}P_{s-K}\right)^{(r)},\ K=r+1,r+2\cdots s\right](1).$$

as the remainder. By giving r all values $0, 1, 2, 3 \cdots$ (10) expresses the whole of R_2.

$$R_\alpha = P_2\Big[(s-3)a_{33}\theta_{s-3}+a_{44}\{(s-4)\theta_{s-4}+(2s-7)\theta'_{s-4}\}+\ \text{etc}\qquad\Big]$$

(a) $\quad +\frac{n+1}{4}\Big[a_{22}\theta_{s-2}+a_{33}\theta'_{s-3}+a_{44}\theta''_{s-4}+[illegible]\qquad\Big]$

$$C_2 = (CP_2+\frac{n+1}{4})\Big[[illegible]\Big]$$
$$+\ P_2\sum_{K=1}^{K=s-2}\Big[a_{KK}\sum_{m=1}^{m=K-4}[illegible]\qquad r=K-3\Big]$$

etc. etc. In R_2 suppose the coefficient of P^{n}_{s-n} is $A(n+1)+B+C$, then A is obtained from $(-1)^r\frac{1}{2}\binom{s}{2}\binom{s-2}{r-1}\binom{2s-r-2}{2s-3}\cdot\binom{s-2}{K-2}K\,2!\,\left(\mu^{K+1}P_{s-K}\right)^{(r)}\ r=n, n+1\cdots n+1$

B from $(-1)^r\frac{1}{2}\binom{s}{2}\binom{s-2}{r-1}\binom{2s-r-2}{2s-3}\cdot\binom{s-2}{K-r+1}\left(\mu^{K+1}P_{s-K}\right)^{r}\quad r=n, n+1,\cdots K$

and C from $(-1)^K\binom{s}{K}\binom{[illegible]}{K-1}\binom{[illegible]}{2s-3}\cdot\left\{\binom{K}{i}(s-n)+\binom{K}{i-1}\right\}$ where $\binom{2s-K-2}{[illegible]}$ is the coefficient of the binomial coefficient $\binom{[illegible]}{2s-n-2}$. Expanding the first of these expressions

$$A = (-1)^{v}\binom{s}{k}\binom{k}{2s-3}\binom{s-2}{v-1}\frac{1}{4v}\Big[\frac{2s-v-2 \cdots\cdots 2s-k-2}{k-v+1!}\,k-v-1 - \frac{2s-v-3 \cdots\cdots 2s-k-2}{k-v!}\times$$

$$k-v-2\,\frac{s-v-1}{1!} + \frac{2s-v-4 \cdots 2s-k-2}{k-v-1}\,k-v-3\,\frac{s-v-1.s-v-2}{2} - + - + \cdots$$

$$(-1)^{k-v}\,\frac{2s-k.\,2s-k-1 \cdots 2s-k-2}{3!}\times\frac{s-v-1 \cdots\cdots s-k+2}{k-v-2!} \pm \cdots\Big]$$

$$= (-1)^{v}\binom{s}{k}\binom{k}{2s-3}\binom{s-2}{v-1}\frac{1}{4v}\Big[2s-k-2\Big(\frac{s-k+v \cdots s-1}{k-v!} \mp \frac{s-v-1 \cdots s-k}{k-v!}\Big)$$

$$-2\Big(\frac{s-k+v-1 \cdots s-1}{s-v+1!} \mp \frac{s-v-1 \cdots s-k-1}{k-v-1!}\Big)$$

Use upper or lower signs according as $k-v$ is odd or even. To obtain this result;

expand

(a) $$x^2(1-x)^{s-v-1} = x^2 - (s-v-1)x^3 + \frac{s-v-1\,s-v-2}{2}x^4 - + \mp (-1)^{k-v}\binom{s-v-1}{k-v-2}x^{k-v}$$

and $$x^{-2}(1-x)^{-(2s-k-2)} = \frac{1}{x^2} + (2s-k-2)\frac{1}{x} + \frac{2s-k-2.\,2s-k-1}{2} + \frac{2s-k \cdots}{3!\,2s-k-3!}\,x + + + \cdots$$

$$+ \frac{2s-v-2!}{k-v+1!\,2s-k-3!}\,x^{k-v-1} + \cdots$$

then differentiate

$$-2x^{-3}(1-x)^{-(2s-k-2)} + (2s-k-2)x^{-2}(1-x)^{-(2s-k-1)} = \frac{-1}{x^3} + \frac{2s-k-2}{x^2} + 0 + \frac{2s-k \cdots 1}{3!\,2s-k-3!}$$

(b) $$+ + + k-v-1.\frac{2s-v-2!}{k-v+1!\,2s-k-3!}\,x^{k-v-2}$$

The coefficient of x^{k-v} in the product of the right members of (a) and (b) is the series of terms in square parenthesis and the coefficient of x^{k-v} in the product of the left members is the quantity within square parenthesis in the final value given for A.

B. is formed by putting $(1-x)^{s-v-1}$ and

$(1-x)^{2s-k-2}$ equal to their expansions, and taking the coefficient of $x^{[illegible]+1}$ [illegible] of the [illegible] [illegible] the [illegible] of the right member. Then

$[illegible] = (-1)^{[illegible]} \frac{s}{k} [illegible] \left[\frac{s-1 \cdots s-k+v-1}{k-v+1} \mp \frac{s-v-1!}{k-v+1!\,s-k-2!} \cdots \pm \frac{s-v-1!\,2s-k-[illegible]}{k\;v!\,s-k-1!} \right]$

When in these expressions for A, B and C, v is made zero, for all odd values of k $\quad A = 0 = B + C$ (11)

For $v = [illegible]$

k even

$A(n+1) + B + C = [illegible] \quad \frac{[illegible]}{k \cdot k+1}$ (12)

For $v = 1$ and k increased by unity we get a value for $A(n+1) + B + C$ which is the preceding value multiplied by $-\frac{s-k}{2}$. Then

[illegible] be the coefficient [illegible] P_n [illegible]

$-\frac{s}{2}k$ [illegible] is the coefficient of P''_{s-n} [illegible] (13)

When $v = k-2$, let $A(n+1) + B + C$ be denoted by α_{1k}; the following are the values of A, B and C when

$v = K - 2$

$$A = (-1)^K \binom{S}{K}\binom{S-2}{K-1}\binom{K}{2S-3} \frac{K-1 \cdot 2S-K-2 \cdot 2S \ldots K\,1 \ldots}{S-K+1 \cdot S-K \cdot 4}$$

$$B = (-1)^K \binom{S}{K}\binom{S-2}{K-1}\binom{K}{2S-3} \frac{2S-K-1 \cdot 2S-K-2 \cdot 2S-2K+3 \cdot K-1}{2 \cdot 6}$$

$$C = (-1)^{K+1} \binom{S}{K}\binom{S-2}{K-1}\binom{K}{\ldots} \frac{2S-K-2 \cdot 3S-2K-2 \cdot K-1}{2 \cdot 6},$$

Now

the coefficient of any of the P_s^v must equal zero, when the whole remainder is considered. Let us take those of dimension $s-2$. They will be found in RL. and in ω of R_x (pp 5) only. The coefficient of P_{s-2} is

$$-\alpha_{12} + \frac{n+1}{6} a_{22} \text{ which equals zero.}$$

And when $v = 0$ & $K = 2$ $\alpha_{12} = \frac{1}{6}\binom{S}{2}\binom{S-2}{1}\binom{2}{2S-3}\left(2S-3 \cdot n+1 + S^2 - 5S + 6\right)$

Therefore $a_{22} = -\frac{1}{n+1}\binom{S}{2}\binom{S-2}{1}\binom{2}{2S-3}\left(\ldots 3 \cdot n+1 + S^2 \ldots\right)$

The coefficient of P_{s-3}' is, by (13)

$$\frac{n+1}{6} a_{33} + \frac{n+1}{6} a_{22} \frac{S-2}{2} - \frac{S-2}{2} \alpha_{12}$$

$= \frac{n+1}{6} a_{33}$; the last two terms cancel by what precedes, and as the whole coefficient $= 0$

$$a_{33} = 0$$

The coefficient of P_{s-4}'' is $\frac{n+1}{6} a_{44} + \frac{n+1}{6} a_{22} \binom{S-2}{3} \frac{1}{4 \cdot 2 \ldots 3} \ldots \alpha_{14}$ $= 0$ Reducing this gives

$$a_{44} = \frac{4}{n+1}\binom{s}{4}\left(\frac{s-2}{2}\right)\frac{2s-8!}{2s-3!}\left\{2(n+1)2s-5+(s-4)(s-5)\right\}$$

Calling the three terms whose sum vanishes and forms the coefficient of P''_{s-4} λ, μ, ν; then the coefficient of $P^{(3)}_{s-5}$ is

$$\frac{n+1}{6}a_{55} + \frac{s-4}{2}\lambda + \frac{s-4 \cdot s-5}{3 \cdot 2s-8}\mu + \alpha_{15}$$

the last three terms reduce to zero; therefore,

$$a_{55} = 0$$

Calling these four terms $0 \cdot \lambda_1\ \mu_1\ \nu_1$

the coefficient of $P^{(4)}_{s-6} = \sigma_2 + \lambda_2 + \mu_2 + \alpha_{16}$, say, =

$$\frac{n+1}{6}a_{66} + \frac{s-5 \cdot s-6}{2 \cdot 2s-11}\lambda_1 + \frac{s-5 \cdot s-6}{4 \cdot 2s-9}\mu_1 + \alpha_{16} = 0$$

Reducing this

$$a_{66} = -\frac{6}{n+1}\frac{s!\,s-2!\,2s-12!}{s-6!\,s-6!\,2s-3!\,2 \cdot 3!}\left\{3(n+1)2s-7 + s-6 \cdot s-7\right\}$$

Similarly a_{77} may be shown to equal zero; and $a_{88} = \frac{6}{n+1}\frac{s!\,s-2!\,2s-16!}{2 \cdot 3!\,s-8!\,s-8!\,2s-3!}\left\{4(n+1)2s-9+s-8 \cdot s-9\right\}$.

Here the terms in the coefficient of $P^{(5)}_{s-7}$ being denoted by $\frac{n+1}{6}a_{77}$ σ_3 $\lambda_3 \cdot \mu_3$ and α_{17} those giving

a_{88} would be $\frac{n+1}{6}a_{88} + \frac{s-7 \cdot s-8}{2 \cdot 2s-15}\sigma_3 + \frac{s-7 \cdot s-8}{4 \cdot 2s-13}\lambda_3 + \frac{s-7 \cdot s-8}{6 \cdot 2s-11}\mu_3 + \alpha_{18}$

It thus appears that $\lambda_3 \cdot \mu_3$ and σ_3 are the similar multiples of λ_2, μ_2 and σ_2; and these are the similar multiples of λ_1 μ_1 and σ_1, etc

Following the same law would give for the coefficient of $l_K^{(K-2)}$

$$\frac{n+1}{6} a_{KK} - \alpha_{1K} + (-1)^{K+1}\left[\sum \frac{s!\,(s-2)!\,(2s-K-2p-2)!}{4!\,(K-2p)!\,(s-K)!\,(s-K+1)!\,(2s-3)!}\, 2s-2K \cdot 2s-2K-2 \cdot 2s-4p-1 \cdot \theta(p)\right.$$

$$\text{where } \theta(p) \equiv \left(p(n+1)(2s-2p+1) + (s-2p)(s-2p-1)\right),\ 2p = 2.4.6 \cdots K \text{ or } K-1\Big]$$

and when K is odd $a_{KK} = 0$, when it is [illegible]

$$a_{KK} = \frac{6}{n+1} \frac{s!\,(s-2)!\,(2s-2K)!}{4.3!\,(s-K)!\,(s-K)!\,(2s-3)!}\left(\tfrac{K}{2}(n+1)(2s-K+1) + (s-K)(s-K-1)\right)$$

To prove this law consider the series $\Gamma \equiv$

$$(-1)^K\Big\{(n+1)\,\frac{[illegible]}{[illegible]\,(2s-3)!} - \frac{s!\,(s-2)!\,(2s-K-1)!\,(2s-2K)(s-K+1)}{4!\,(s-K)!\,(s-K+1)!\,(K-2)!\,(2s-3)!}$$

$$+ \frac{s!\,(s-2)!\,(2s-K-2)!\,(3s-2K-2)(2s-2K)(s-K+1)}{4!\,(s-K)!\,(s-K+1)!\,(K-2)!\,(2s-3)!}$$

$$- \sum \frac{s!\,(s-2)!\,(2s-K-2p-2)!}{4!\,(K-2p)!\,(s-K)!\,(s-K+1)!\,(2s-3)!}\, 2s-2K \cdot 2s-2K+2 \cdot 2s-4p-1 \cdot \theta(p)$$

$$2p = 2.4.6 \cdots K-1 \text{ or } K$$

The first three terms are what A, B and C become when v is put equal to $K-2$.

The series is to be shown to be equal to zero.

The common factor, $(-1)^K \dfrac{s!\,(s-2)!\,(2s-2K-2)!}{4!\,(s-K)!\,(s-K+1)!\,(2s-3)!}$ may be omitted and writing $\chi(q)$ for

$$\frac{2s-K-q \cdot 2s-K-q-1 \cdot 2s-K-q-2 \cdots\cdots 2s-2K-1}{(K-q+2)!},$$

$$\chi(q) = \frac{2s-K-q \cdot 2s-K-q-1}{K-q+2 \cdot K-q+1}\ \chi(q+2) \quad . \qquad (14)$$

The series to be considered becomes

$X(4) \cdot 2s-K \cdot 2s-K-1 \cdot 2s-K-2 \cdot 2s-K-3 \cdot (n+1)$

$- X(4) \cdot 2s-K-1 \cdot 2s-K-2 \cdot 2s-K-3 \cdot 2s-2K \cdot s-2K+3$

$+ X(4) \cdot 2s-K-2 \cdot 2s-K-3 \cdot 2s-2K \cdot s-K+1 \cdot 3s-2K-2$

$- X(4) \cdot 2s-2K \cdot 2s-2K+2 \cdot 2s-5 \{2n(2s-3) + s^2 - 3s + 3\}$

$- X(6) \cdot 2s-2K \cdot 2s-2K+2 \cdot 2s-[illegible] (2n \cdot 2s-5 + s^2 - 5s + 10)$

$- X([illegible]) \cdot [illegible]-2K \cdot 2s-2K+2 \cdot 2s-[illegible] \cdot (2n \cdot 2s-7 + s^2 - [illegible] + 21)$

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$- X(2p+2) \cdot 2s-2K \cdot 2s-2K+2 \cdot 2s-4p-1 \left(pn(2s-2p-1) + s^2 - (2p+1)s + p(2p+1)\right)$

$$2p = K-1 \text{ or } K$$

Omitting the part independent of n,

$X(4) [2s-K \cdot 2s-K-1 \cdot 2s-K-2 \cdot 2s-K-3 - 2s-2K \cdot 2s-2K+2 \cdot 2s-5 \cdot 2s-3] =$

$X(4) [8s^2 - 4s(2K+3) + K(K+11)] K-2 \cdot K-3 = K-2 \cdot K-3 \, X(4) \Delta_1$ say,

$= 2s-K-4 \cdot 2s-K-5 \, \Delta_1 \, X(6)$, by (4). Take from this

$X(6) \, 2s-2K \cdot 2s-2K+2 \cdot 2s-9 \cdot 2s-5 \cdot 2$ and the second remainder is $X(6) K-4 \cdot K-5 [12s^2 - 6s(2K+5) + K(K+29)] \equiv$

$K-4 \cdot K-5 \, X(6) \Delta_2$ suppose. This equals

$2s-K-6 \cdot 2s-K-7 \, X(8) \Delta_2$ by (4). Take from this the next term $X(8) \, 2s-2K \cdot 2s-K+2 \cdot 2s-13 \cdot 2s-7 \cdot 3$. The remainder is $K-6 \cdot K-7 \, X(8) [16s^2 - 8s(2K+7) + K(K+55)]$

$= 2s-K-8 \cdot 2s-K-9 \cdot X(10) \Delta_3$ say, as above,

Supposing this [illegible] for the differences till the (m-1)th, then it holds for the mth. The (m-1)th is

$$\overline{K+2-2m}\,.\,\overline{K+1-2m}\ X(2m)\left[4ms^2-2ms\,(2K+2m-1)+K(K+4m^2-2m-1)\right]$$

$$= X(2m+2)\,\Delta_{(m-1)}\ \overline{2s-K-2m}\,.\,\overline{2s-K-2m-1}.$$ Taking from this [illegible] the remainder

$$[illegible]\ \overline{K-2m}\,.\,\overline{K-2m-1}\ X(2m+2)\ [illegible]\ (2K+2m+1))+K(K+4m^2+2m-1))$$

$$= \overline{K-2m}\,.\,\overline{K-2m-1}\,.\,X(2m+2)\,\Delta_{(m)}.$$ That is, the mth, difference is the same function of m as the $(m-1)$th is of $m-1$,

When $2m=2p=K-1$ or K the subtrahend is the last term of the series, and the difference vanishes. Therefore the coefficient of x in the series vanishes.

The algebraic sum of the first four terms independent of m is

$$\overline{K-2}\,.\,\overline{K-3}\ X(4)\left[2s^3\ [illegible]\right] - \overline{K-2}\,.\,\overline{K-3}\ X(4)\,\Delta_1,$$ say, then it equals, by (14)

$$\Delta_1\,,\ \overline{2s-K-4}\,,\ \overline{2s-K-5}\ X(6).$$ Taking from this

$\chi(6)\, 2s-2k \cdot 2s-2n+2 \cdot 2s-7 \cdot s^2 - 5s + 10$, there remains

$\chi(6)\, k-4 \cdot k-5 \{2s^3 - (2n+12)s^2 + (4k+28)s - n(k+2)\} =$

$\chi(6)\, k-4 \cdot k-5 \cdot \Delta_2$ say. If the $(m-1)$th difference be

$$k-2m+2 \cdot k-2m+1\ \chi(2m)\left[2s^3-(2n+4m)s^2+\{2n(2m+1)+2(m-1)(2m+1)\}s - n(n+4m^2-2m-1)\right] \equiv \psi(m-1),$$

then the mth is $\psi(m)$. For taking

$$\chi(2m+2)\, 2s-2k \cdot 2s-2k+2 \cdot 2s-4m-1\{m\,n(2s-2m-1)+s^2-(2m+1)s+m(2m+1)\}$$

from the $(m-1)$th remainder, there remains

$$\chi(2m+2)\, k-2m \cdot k-2m-1\left[2s^3-s^2(2n+4(m+1)) + \{2n(2m+3)+2m(2m+3)\}s - n(n+4m^2+6m+1)\right] \equiv \psi(m) \qquad Q.E.D.$$

Zero remainder when $2m = k-1$ or k. whatever

[illegible]

[illegible]

seen that both the coefficient of n and the part independent of n separately vanish in the series considered for all values of k. (15)

Assuming that $a_{kk} = 0$ when k is odd, and $= -\dfrac{6}{n+1}\dfrac{s!\,s-2!\,2s-2k!}{s-k!\,s-k!\,2s-3!\,2\cdot 3!}\left\{\frac{k}{2}(n+1)(2s-k-1)+(s-k)(s-k-1)\right\}$

for all values of $K \overline{\gtrless} 2w$. then it may be shown to be true when $K = 2w+1$ and $2w+2$.

The coefficient of $P^{(2w-1)}_{s-2w-1}$ in RL is $\alpha_{1,2w+1}$. and, if M^s_k represent the value of a_{KK} when K is even and N^T_s represent the expression $(-1)^T \frac{s!\,s-2!\,2s-T-2!}{2!\,T!\,s-T!\,s-T-1!\,2s-3!}$, i.e. the coefficient of $P^{(T)}_{s-T}$ in L. then the whole coefficient of $P^{(2w-1)}_{s-2w-1}$ is

$$\tfrac{n+1}{6}a_{2w+1,2w+1} + \alpha_{1,2w+1} + \left[M^s_2 N^{2w-1}_{s-2} + M^s_4 N^{2w-3}_{s-4} + M^s_6 N^{2w-5}_{s-6} + \cdots + M^s_{2w} N^1_{s-2w}\right]\left[-\frac{n+1}{6}\right] = 0$$

Now $\alpha_{1,2w+1}$ is the first three terms of Γ and the following terms are those of Γ also for taking any one of them as $M^s_{2z} N^{2w-2z+1}_{s-2z}$, $z \overline{\gtrless} w$, it becomes when written

in L [illegible] $\frac{+6(n+1)s!\,s-2!\,2s-z!}{n+1.6\,s-2z!\,s-2z!\,2s-3!\,3!\,2}\{2(n+1)\ldots - 2\} + (n-2)(\ldots 1)\}$ ×

$$\frac{s-2z!\,s-2z-2!\,2s-2w-2z-3}{2!\,2w-2z+1!\,s-2w-1!\,s-2w-2!\,2s-4z-3!}$$

$$= \frac{s!\,s-2!\,2s-2w-2z-3!\,f(z)}{4!\,2w-2z+1!\,s-2w-1!\,s-2w!\,2s-3!} \cdot 2s-4w \cdot 2s-4w+2 \cdot 2s-4z-1$$

which coincides with the last terms of Γ, when $K = 2w+1$, and $z = p$

Thus the coefficient of P^{2w+1}_{s-2w-1} consists of $a_{2w+1,2w+1}\frac{n+1}{6}$ plus a series of terms which vanish (15). Therefore

$$a_{2w+1,2w+1} = 0 \qquad (16)$$

The coefficient of P^{2w+2}_{s-2w-2} is

$$\frac{n+1}{6}a_{2w+2,2w+2} + \alpha_{1,2w+2} + \frac{n+1}{6}\Big[\mathcal{M}^S_2 N^{2w}_{s-2} + \mathcal{M}^S_4 N^{2w-2}_{s-4} + +$$
$$+ + \mathcal{M}^S_{2w} N^{2}_{s-2w}\Big] = 0$$

Γ also gives all the terms in this expression when $K = 2w+2$, excepting the first, or $a_{2w+2,2w+2}\frac{n+1}{6}$. But the last term $\mathcal{M}^S_{2w} N^2_{s-2w}$ is the second last in Γ when $K = 2w+2$; $2p = 2, 4, \dots 2w+2$.

Taking Γ from the above coefficient

$\frac{n+1}{6}a_{2w+2,2w+2} - \frac{n+1}{6}\mathcal{M}^S_{2w+2}$ is [illegible] obtained, since $\Gamma = 0$ always, and as this must vanish

$$a_{2w+2,2w+2} = \mathcal{M}^S_{2w+2} \qquad (17)$$

Thus (16) shows that if for any odd value of K and all lower values $a_{KK} = 0$, then $a_{K+2,K+2} = 0$ [illegible] [illegible] values $a_{KK} = \mathcal{M}^S_K$, then $a_{K+2,K+2} = \mathcal{M}^S_{K+2}$, but on pps 9 and 15 it is

shown that $a_{KK} = 0$ for $K = 3, 5, 7,$ and $a_{KK} = M^S_K$ for $K = 2, 4, 6, 8 \ldots$. Therefore, it follows that (16) and (17) are true for all values of w. Then, in θ_s, the row of terms designated α and of which R_2 is a factor, contains no invariant or derivative of form $\theta^{(2u-1)}_{s-2u-1} \ldots$ (18)

This is also the case for the terms entering in the row designated β and of which R_2' is a factor. For the term $R_2' \theta'_{s-4}$ is found only in $R\alpha$ and $R\beta$. Its coefficient is $+2b_{44} + (2s-7)a_{44}$ then

$$b_{44} = -\frac{2s-7}{2} a_{44}$$

Any term as $R_2 \theta^{(K-3)}_{s-K}$ K being odd could appear only in $R\alpha$ and $R\beta$, ~~and~~ as it does not appear in $R\alpha$, it cannot in $R\beta$. The coefficient of $R_2 \theta^{(2u-3)}_{s-2u}$ is

$$+2b_{2u\,2u} + \left(\binom{2u-2}{2u-3}(s-2u) + \binom{2u-2}{2u-4}\right)a_{2u,2u} = 0$$

$$\therefore b_{2u\,2u} + (u-1)(2s-3-2u)a_{2u,\,2u} \qquad - \qquad (19)$$

The terms of dimension $s-1$ and form $R_2' \theta^{(K-4)}_{s-K}$

can appear only in R_β and R_γ and when κ is odd no such term appears in R_β, therefore it does not enter into R_γ and when κ is even the coefficient of $P_2' \theta_{5-2u}^{2u-4}$ is

$$5c_{2u2u} + \left\{ \binom{2u-3}{2u-4}(5-2u) + \binom{2u-3}{2u-5} \right\} b_{2u,2u} = 0$$

$$5c_{2u,2u} + (2u-3)(5-u-2) b_{2u,2u} = 0 \qquad (20)$$

Thus it is easy to see, by taking one row after another, that the non-linear part of θ_5 contains no term having $\theta_{5-\kappa}^{(\nu)}$ as a factor when κ is odd. (21)

From this it follows that if all the invariants of a differential equation with even suffix vanish the linear part of each vanishes. The same is true for those with odd suffix (22)

Section VI

The coefficients of θ_s

θ_s has, as we have seen, a linear part expressed by $\sum_{T=0}^{T=s-2} N_s^T \beta_{s-T}^{(T)}$ or

$$\sum_{T=0}^{T=s-2} (-1)^T \frac{s!\,(s-2)!\,(2s-T-2)!}{2\cdot T!\ [illegible]\ T!\,(T+1)!\ [illegible]} \beta_{s-T}^{(T)} \qquad (23)$$

Then follow a series of terms

$\beta_2 \{ a_2 \theta_{s-2} + a_4 \theta''_{s-4} + a_6 \theta^{iv}_{s-6} + a_8 \theta^{v}_{s-8} + + +$ etc $\}$ expressed

by [illegible] $- \frac{[illegible]}{[illegible]} \beta_2 \sum_{K=1}^{K=\left[\frac{s-2}{2}\right]} \Big[$ [illegible] $\Big] \theta^{2K-1}_{s-2K}$ (24)

The [illegible] coefficients [illegible] [illegible] the proof not given here and thus one may [illegible] [illegible] terms

like $\beta_2' [b_4 \theta'_{s-4} + b_6 \theta'''_{s-6} + +]$

$+ \beta_2'' [c_4 \theta_{s-4} + c_6 \theta''_{s-6} + +]$

$+ \beta_2''' [e_6 \theta'_{s-6} + e_8 \theta'''_{s-8} + +]$

$+ \beta_2^{iv} [g_6 \theta_{s-6} + g_8 \theta''_{s-8} + +]$

$+ \beta_2^{(5)} [l_8 \theta'_{s-8} + l_{10} \theta'''_{s-10} + +]$

[illegible]

$$P_2^{(\nu)} \sum_{K=\frac{\nu+2}{2}}^{K=\left[\frac{s-3}{2}\right]} n_{2K}\, \theta_{s-2K}^{(2K-\nu-2)}$$ When ν is even.

$$+\, P_2^{(\nu+1)} \sum_{K=\frac{\nu+4}{2}}^{K=\left[\frac{s-3}{2}\right]} p_{2K}\, \theta_{s-2K}^{(2K-\nu-3)}$$ $\nu = 0, 2, 4, 6$ — — —

$\left[\frac{s-3}{2}\right]$ is the greatest integer in $\frac{s-3}{2}$.

If any two consecutive rows ($\nu = \mu$) [illegible] arising from them will contain a term, — $P_2^{(\mu)}\theta_{s-2K}^{(2K-\mu-3)}$ — found nowhere else, because rows preceding these two have multipliers $P_2^{(\nu)}$, ν being $< \mu$; and rows following them have a remainder in which the index of θ_{s-2K} cannot be as great as $(2K-\mu-3)$.

This remainder is

$$\left[\frac{\mu(\mu+1)}{2!}\,P_2^{(\mu+1)} + [illegible]\,P_2^{(\mu+2)} + [illegible]\,P_2^{(\mu+3)} + \cdots\right] \sum_{K=\frac{\mu}{2}}^{K=\left[\frac{s-3}{2}\right]} n_{2K}\, \theta_{s-2K}^{(2K-\mu-2)}$$

$$+\, P_2^{(\mu)} \sum_{K=\frac{\mu+2}{2}} [illegible] \sum_{m=0} \left\{\binom{[illegible]}{m} + \binom{[illegible]}{m-1}\right\} \theta_{s-2K}^{([illegible])} [illegible]$$

$$+\left\{\frac{[illegible]}{[illegible]\,2!}(\mu+1)P_2^{(\mu)} + \text{terms of lower dimensions}\right\} \sum_{K=\frac{\mu+4}{2}}^{K=\left[\frac{s-3}{2}\right]} p_{2K}\, \theta_{s-2K}^{(2K-\mu-3)}$$

$$+ P_2^{(\mu+1)} \left[\sum_{k=\frac{\mu+4}{2}}^{k=\left[\frac{s-3}{4}\right]} p_{4k} \sum^{m=\nu-1} \left\{ \binom{2}{m}(s-2k) + \binom{2}{m-1} \right\} t_{s-2k}^{(2m)} \right.$$

Equating the coefficient of the term $P_2^{(\mu)} t$ [illegible] we obtain

$$\frac{\mu+1!}{\mu!\,2!}(4+\mu)\,p_{2k} = -\left\{ \binom{2k-\mu-2}{2k-\mu-3}(s-2k) + \binom{2k-\mu-2}{2k-\mu-4} \right\} n. \qquad (25)$$

In this K is any number and μ any of the values of ν so that the coefficients p_{2k} of any row may be expressed in terms of those of the preceding row, viz n_{2k}.

(25) when simplified gives

$$\frac{\mu+1}{2}(4+\mu)\,p_{2k} = -(2k-\mu-2)\left(s-k-\frac{\mu+3}{2}\right) n_{2k}$$
$$= -\frac{(2k-\mu-2)(2s-2k-\mu-3)}{2}\, n_{2k}$$

$\mu=0$ gives. $4.\frac{1}{2}\, b_{2k} = -\frac{(2k-2)(2s-2k-3)}{2}\, a_{2k}$

$\mu=1$ " $5.\frac{2}{2}\, c_{2k} = -\frac{(2k-3)(2s-2k-4)}{2}\, b_{2k}$

$\mu=2$ " $6.\frac{3}{2}\, e_{2k} = -\frac{2k-4}{2}(2s-2k-3)\, c_{2k}$

- - - - " - - - -

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$$\frac{\mu+1}{2}(4+\mu)\,p_{2k} = -\frac{2k-\mu-2}{2}(2s-2k-\mu-3)\, n_{2k}$$

and multiplying the right members of these equations together and also the left and equating the products

$$\frac{\mu+1!\ \mu+4!}{3!}\,p_{2\kappa} = (-1)^{\mu+1}\frac{2\kappa-2!\ 2s-2\kappa-3!}{2\kappa-\mu-3!\ 2s-2\kappa-\mu-4!}\,a_{2\kappa}\ldots \quad (25)$$

The p's being coefficients in the row multiplied by $P_2^{(\mu+1)}$ and $\mu+1$, odd or even according as the number of equations multiplied was odd or even, it is seen that the coefficient of any term of the form $P_2^{(\delta)}\,[illegible]_{s-2\kappa}$ may be expressed in terms of the a's. To facilitate the work the coefficient of such a term will be written $(\delta)^{(2\kappa-\delta-2)}_{2\kappa}$. Then—

$$(\delta)^{(2\kappa-\delta-2)}_{2\kappa} = (-1)^{\delta+1}\frac{2\kappa-2!\ 2s-2\kappa-3!\ 3!}{\delta!\ \delta+3!\ 2\kappa-\delta-2!\ 2s-2\kappa-\delta-3!}\ [illegible]\ \frac{\theta(\kappa)}{3!\,2}\ [illegible]$$

where $\theta(\kappa) = \{\kappa(\kappa+1)(2s-2\kappa-1) + (s-2\kappa)(s-2\kappa-1)\}$ (27)

There still remain terms of the form $P_2^{(\alpha)a}P_2^{(\beta)b}P_2^{(\gamma)c}P_2^{(\delta)d}P_2^{(\epsilon)e}(\ldots)^{2\kappa}_{\kappa}F^{(\kappa)}_{2\kappa}$.

Here a, b, c, d and e are indices expressing power

of the quantities to which they are attached. $(\alpha^a\beta^b\gamma^c\delta^d\varepsilon^e)^m_{2K}$ is the coefficient of the term, having such indices, powers and suffix, $s-2K$, throughout the whole invariant the order of the factors will be taken so that

$$\alpha \geqq \beta \geqq \gamma \geqq \delta \geqq \varepsilon \quad \text{etc.}, \tag{28}$$

$$2K = m + a(\alpha+2) + b(\beta+2) + c(\gamma+2) + d(\delta+2) + e(\varepsilon+2) \tag{29}$$

The coefficient of this term that is, the numerical value of $(\alpha^a\beta^b\gamma^c\delta^d\varepsilon^e)^m_{2K}$ is found by equating the coefficient of $h_2^{(\alpha)a} h_2^{(\beta)b} h_2^{(\gamma)c} h_2^{(\delta)d} h_2^{(\varepsilon)e} t_{s-2K}$ in the remainder to zero. It is

$$\frac{e}{2}(m+1)(\alpha^a\beta^b\gamma^c\delta^d\varepsilon^{e-2})^{(m+1)}_{2K} + (\alpha^{a-1}\beta^b\gamma^c\delta^d\varepsilon^{e-1}\,\varepsilon+\alpha+2)^{(m)}_{2K}\frac{\varepsilon+\alpha+2!}{\alpha!\,\varepsilon+3!}(2\varepsilon+6+\alpha)$$

$$+ (\alpha^a\beta^{b-1}\gamma^c\delta^d\varepsilon^{e-1}\,\varepsilon+\beta+2)^{(m)}_{2K}\frac{\varepsilon+\beta+2!}{\beta!\,\varepsilon+3!}(2\varepsilon+6+\beta)$$

$$+ (\alpha^a\beta^b\gamma^{c-1}\delta^d\varepsilon^{e-1}\,\varepsilon+\gamma+2)^{(m)}_{2K}\frac{\varepsilon+\gamma+2!}{\gamma!\,\varepsilon+3!}(2\varepsilon+6+\gamma)$$

$$+ (\alpha^a\beta^b\gamma^c\delta^{d-1}\varepsilon^{e-1}\,\varepsilon+\delta+2)^{(m)}_{2K}\frac{\varepsilon+\delta+2!}{\delta!\,\varepsilon+3!}(2\varepsilon+6+\delta)$$

$$+ (\alpha^a\beta^b\gamma^c\delta^d\varepsilon^{e-2}\,2\varepsilon+2)^{(m)}_{2K}\frac{2\varepsilon+2!}{\varepsilon!\,\varepsilon+3!}(3\varepsilon+6)$$

$$+ (\alpha^a\beta^b\gamma^c\delta^d\varepsilon^{e-1})^{(2K-m)}_{2K}\frac{m+\varepsilon+2!}{m!\,\varepsilon+3!}\{(\varepsilon+3)(s-2K)+m\}$$

$$+\frac{n+1}{2}\sum(\;)^{\pi}_{n+\varepsilon+2}\sum\left(\alpha_1^a\beta_1^b\gamma_1^c\delta_1^d\varepsilon_1^{e-1}\right)^{(r)}\frac{\pi!}{r!(\pi-r)!}\left(\alpha_1^a\beta_1^b\gamma_1^c\delta_1^d\varepsilon_1^{e-1}\right)\ldots$$

$$=0 \qquad (30)$$

In this $(\varepsilon)^{\pi}_{\pi+\varepsilon+2}$ is the coefficient of $P_2^{(\varepsilon)}\theta^{\pi}_{s-\pi\ldots}$

The values of π are $\pi=0.2.4\ldots\ldots 2K-2(a+b+c+d+e)-\varepsilon$ (31)

if ε is even, and if ε is odd $\pi=1.3.5\ldots\ldots 2K-2(a+b+c+d+e)-\varepsilon$

$\left(\alpha_1^a\beta_1^b\gamma_1^c\delta_1^d\varepsilon_1^{e-1}\right)^{(r)}$ is the numerical coefficient of $\left(P_2^{(\alpha)a}P_2^{(\beta)b}P_2^{(\gamma)c}P_2^{(\delta)d}P_2^{(\varepsilon)e-1}\right)$ in $\frac{d^r}{dr}\left(P_2^{(\alpha_1)a}P_2^{(\beta_1)b}P_2^{(\gamma_1)c}P_2^{(\delta_1)d}P_2^{(\varepsilon_1)e-1}\right)$

$$\alpha_1\overline{\gtrless}\alpha,\ \beta_1\overline{\gtrless}\beta,\ \gamma_1\overline{\gtrless}\gamma,\ \delta_1\overline{\gtrless}\delta,\ \varepsilon_1\overline{\gtrless}\varepsilon. \qquad (32)$$

$y\overline{\gtrless} m$

$r=y+\pi-\ldots$

$\alpha_1\ \beta_1\ \gamma_1\ \delta_1\ \varepsilon_1$ take all possible values consistent with (31) and the ...

the corresponding y is found from

$$y=2K-2(a+b+c+d+e)-\varepsilon-\pi-a\alpha_1-b\beta_1-c\gamma_1-d\delta_1-(e-1)\varepsilon_1$$

In the coefficient $\left(\alpha_1^a\beta_1^b\gamma_1^c\delta_1^d\varepsilon_1^{e-1}\right)\ldots$ to be changed to $s-\varepsilon-2-\pi$

The following examples will illustrate the subject.

First: What is the coefficient of $P_2^{6}\theta^{4K-12}_{s-2K}$

In this $a=b=c=d=e=0$

$a = 0$ $\pi = 0, 2, 4 \ldots 2\kappa-12$, $c = 0$

$u = 2\kappa - 12 - \pi$. Then

$$\frac{6}{6}(n+1)(0^6)_{2\kappa}^{(2\kappa-12)} + 0 + 0 + 0 + 0 + (0^4 2)_{2\kappa}^{2\kappa-12}\frac{6}{3} + (0^5)_{2\kappa}^{(12)}\frac{2\kappa-10\cdot 2\kappa-11}{6}(35-4\kappa-12)$$

$$+\frac{n+1}{6}\left[(0)_{2}(0^5)_{2\kappa-2}^{(2\kappa-12)} + (0)_4^{(2)}(0^5)_{2\kappa-4}^{(2\kappa-14)} + (0)_6^{(4)}(0^5)_{2\kappa-6}^{(2\kappa-16)} + \ldots + (0)_{2\kappa-10}^{2\kappa-12}(0^5)_{10}\right] = 0$$

This states that

$(n+1)$ times the coefficient of $P_2^5 t_{5-2\kappa}^{2\kappa-12}$ plus twice the coefficient of $P_2^4 P_2'' t_{5-2\kappa}^{2\kappa-12}$ plus $\frac{2\kappa-10\cdot 2\kappa-11}{6}(35-4\kappa-12)$ times the coefficient of $P_2^5 t_{5-2\kappa}^{12}$ plus $\frac{n+1}{6}$ times a number of terms of the same form $= 0$

Any one of these terms, as, $(0)_6^4(0^5)_{2\kappa-6}^{(2\kappa-16)}$ is written in full thus

The coefficient of $P_2 t_{5-6}^{(4)}$ times the coefficient of $P_2^5 t_{5-2\kappa+6}^{(2\kappa-16)}$ in the invariant t_{5-6}^4.

This last coefficient being known for t_5 all that would be necessary is to change 5 to ...

For another example take $P_2^2 P_2^{(1)3} P_2^{(3)2} t_{5-2\kappa}^{(m)}$ and find the coefficient.

here $2\kappa = m + 2\cdot 3$ $a=2$ $b=3$ $c=2$

[illegible]

$$\frac{2}{6}(n+1)(0^2/^3 3^{2(m)}_{/2K} + (0^2/^3 3.5)^{(m)}_{/2K}(2\cdot 6+0) + (0^2,{}^2 3\,6)^{(m)}_{/2K}\frac{6!}{1!6!}(2\cdot 6+1)$$

$$+ \text{[illegible]}$$

$$+\frac{n+1}{6}\binom{3}{6}\Big\{(0^2/^3 3)^{(m-1)}_{2K-6} + \binom{1}{2} b_2 (0^3/^2 3)^{(m)}_{2,K-6} + (0^2/^3 2)^{(m)}_{/2K-6}\Big\}$$

$$+ \text{[illegible]} + \frac{1}{2} b_2 \text{[illegible]}\, 3)^{m-2}_{/2K-8} + \binom{3}{2} b_2 (\text{[illegible]}\, 2)^{m-2}_{/2K-8} + \frac{1}{2} b_2 \text{[illegible]}$$

$$+\binom{3}{2}(0^3/^2 2)^{(m-1)}_{2K-8} + \binom{3}{2} b_2 (0^2/^4)^{(m-1)}_{/2K-8} + \binom{3}{3} b_2 (0^5 3)^{(m)}_{2K-8} + \binom{3}{3} b_2 (0^4/2)^{(m)}_{2K-8} + \binom{3}{3} b_2 (0^3/^3)^{(m+1)}_{2K-8}\Big\}$$

$$+ \binom{5}{\cdot}\Big\{(1^2\,3\ldots)^{(m-5)}_{2K-10} + \binom{5}{2} b_2 (0^3/^2 3)^{(m-4)}_{2K-10} + \binom{5}{2} b_2 (0^2/^3 2)^{(m-4)}_{2K-10} + \binom{5}{2} b_2 (0^4/3)^{(m-3)}_{2K-10} + \binom{5}{3} b_2 (0^3/^2 2)^{(m-\cdot)}_{2K-10}$$

$$+\frac{5}{2} b_2 (0^2/^4)^{m-3}_{2K-10} + \binom{5}{2} b_2 (0^5 3)^{(m-2)}_{2K-10} + \binom{5}{2} b_2 (0^4/2)^{(m-2)}_{2K-10} + \binom{5}{3} b_2 (0^3/^3)^{(m-2)}_{2K-10} + \binom{5}{3} b_2 (0^5 2)^{(m-1)}_{2K-10}$$

$$+\binom{5}{4} b_2 (0^4/)^{2(m-1)}_{2K-10} + \binom{5}{4} b_2 (0^5/)^{(m)}_{2K-10}\Big\}$$

$$+ \binom{7}{\cdot}\Big\{ \text{[illegible]}^{m-7} + \frac{7}{2} \text{[illegible]}^{m-6} + \frac{7}{2} b_2 \text{[illegible]}^{m-6} + \frac{7}{2} b_2 \text{[illegible]} + \frac{7}{2} b_2 (0^3/^2 2)^{m-5}_{2K-12}$$

$$+\binom{7}{4} b_2 (0^2/^4)^{(m-5)}_{2K-12} + \binom{7}{2} b_2 (0^5 3)^{(m-4)}_{2K-12} + \binom{7}{2} b_2 (0^4/2)^{(m-4)}_{2K-12} + (0^3/^3)^{(m-4)}_{2K-12}$$

$$\binom{7}{2} b_2 (0^5 2)^{(m-3)}_{2K-12} + \binom{7}{2} b_2 (0^4/)^{2(m-3)}_{2K-12} + \binom{7}{2} b_2 (0^5/)^{m-2}_{2K-12} + \binom{7}{4} b_2 (0^6)^{(m-1)}_{2K-12}\Big\}$$

$$+\binom{\cdot}{\cdot}^{(9)}\Big\{(0^2/^3 3)^{(m-9)} + \binom{9}{2} b_2 (0^3,2\,2)^{(m-8)} + \binom{9}{2} b_2 (0^2/^3 2)^{(m-8)} + \frac{9}{2} b_2 (0^4 \cdot)^{m-7} + \cdots$$

$$- - \binom{\cdot}{2} b_2 (0^5 2)^{(m-5)} + \binom{9}{2} b_2 (0^4/)^{2(m-5)} + \binom{9}{2} b_2 (0^5/)^{(m-4)} + \binom{9}{2} b_2 (0^6)^{(m-3)}$$

$$+ \quad - - - -$$

$$-$$

$$+$$

$$+ B^{(2K-18)}_{2n-13}\{(0^5/)^{(2)} + (0^6)^{(1)}\}$$

$$+ (\text{[illegible]}) (0^6) \} = 0 \;\text{[illegible]}$$

in this $r = \mu + \pi - m$ always and $\ker$ is the coefficient of $R_2^2 R_2^{(1)3} R_2^{(3)}$ in the term which it precedes. Take the term $(3)_{14}^{9} \binom{9}{r} \ker (0^5 2)_{2K+4}^{(m-5)}$ as illustrating [illegible] means the coefficient of [illegible] $R_2^{(3)} \ldots^{(9)}$ [illegible] the coefficient of $R_2^5 R_2^{\prime\prime} \theta_{5-2K+4}^{(m-5)}$ in the invariant $t_{5-14}^{(9)}$ multiplied by $\binom{9}{r}\ker$. here $r = m-5+9-m = 4$.

$\ker$ = the coefficient of $R_2^2 R_2^{(1)3} R_2^{(3)}$ in

$$\frac{d^4}{dx^4}\left(R_2^5 R_2^{(2)}\right) \quad \text{and} \quad \binom{9}{r} = \frac{9!}{4!\,5!}$$

Thus every term in the invariant θ_s has been considered and by (23), (24) and (27) the classes of terms represented have their coefficients expressed in terms of s and n the order of the differential equation and in (30) in terms of preceding coefficients.

[illegible]

(33) could be expressed much more simply.

[illegible] in terms of s and n [illegible]

Section III

Associate Equations and Associate Variables

In the memoir previously referred to, [illegible] shows that in connection with any differential equation of order n, there are $n-2$ other equations of different orders which he calls associate equations. If the equation be

$$u^{(n)} + \alpha u^{(n-1)} + \beta u^{(n-2)} + \cdots + \mu u' + \nu u = 0 \quad , (34)$$

and its solutions are $u_1\, u_2\, u_3 \ldots\, u_n$ then $\begin{vmatrix} u_a & u_b \\ u_a' & u_b' \end{vmatrix}$, where u_a and u_b are any two of the n solutions,

is a solution of one of the associate equations called the first, and $\begin{vmatrix} u_a & u_b \\ u_a' & u_b' \end{vmatrix}$ the first associate variable.

Similarly the second, third, fourth ..... $n-1$, associate variables are

$$\begin{vmatrix} u_a & u_b & u_c \\ u_a' & u_b' & u_c' \\ u_a'' & u_b'' & u_c'' \end{vmatrix}, \quad \begin{vmatrix} u_a & u_b & u_c & u_d \\ u_a' & u_b' & u_c' & u_d' \\ u_a'' & u_b'' & u_c'' & u_d'' \\ u_a''' & u_b''' & u_c''' & u_d''' \end{vmatrix}, \quad \begin{vmatrix} u_2 & u_4 & u_5 & u_7 & u_9 \\ u_2' & u_4' & u_5' & u_7' & u_9' \\ u_2'' & u_4'' & u_5'' & u_7'' & u_9'' \\ u_2''' & u_4''' & u_5''' & u_7''' & u_9''' \\ u_2^{iv} & u_4^{iv} & u_5^{iv} & u_7^{iv} & u_9^{iv} \end{vmatrix}$$

$$\text{-- -- --} \qquad \begin{vmatrix} u_1 & u_2 & \cdots & u_{n-1} \\ u_1' & u_2' & \cdots & u_{n-1}' \\ - & - & & - \\ - & & & \\ u_1^{(n-2)} & u_2^{(n-2)} & \cdots & u_{n-1}^{(n-2)} \end{vmatrix}$$

where the u's are any combinations of the n variables u. These variables of any one set are particular and linearly independent solutions of an equation. Hence there are connected with the equation $n-2$ other associate equations. The equation satisfied by the n $n-1$ st associate variables is the Lagrangian adjoint equation.

Let u_2 represent the first associate variable which may be written also thus; $(1\,2')$. and e.g. the variable $\begin{vmatrix} u_a & u_b & u_c \\ u_a' & u_b' & u_c' \\ u_a'' & u_b'' & u_c'' \end{vmatrix}$. This may be written $(a\,b'\,c'')$.

then the third associate variable $a_4 \equiv (b\,d'\,c''\,e''')$.

where [illegible] the suffixes of the us. There are then $\binom{n}{3}$ different functions a_3 each of which is a solution of [illegible] ... nation, and likewise $\binom{n}{4}$ different functions a_4. which are solutions of the third associate etc. $a_{n-1} = (u_1 u_2' u_3'' u_4''' \cdots u_{n-1}^{(n-2)})$, only the diagonal of the determinant being written. This may also be written $(1\,2'\,3''\,4''' \quad n-1^{(n-1)})$. as a simple [illegible] ations required. Then $(1\,2'\,3''\,4''' \cdots n^{(n-1)})$ is the non-vanishing constant Δ.

To illustrate what follows I shall [illegible] Then the equation will be.

$$u^{(5)} + 10\varphi_3 u'' + 5\varphi_4 u' + \varphi_5 u = 0 \qquad (34a)$$

u_1, u_2, u_3, u_4, u_5 are the five independent solutions. Then let a_2 represent the determinant [illegible] of any two of these and their first derivatives and as we are not concerned which two

the suffixes need not be expressed, so that a_2 may be written $\overline{01}$, 0 and 1 being the indices of the diagonal of the determinant. Differentiating this, $\frac{da_2}{dx}$ or $a_2' = \overline{02}$

$$a_2'' = \overline{03} + \overline{12}$$

$$a_2''' = [illegible] \overline{04} + [illegible]$$

$$a_2^{iv} = 3\cdot\overline{14} + 2\cdot\overline{23} + \overline{05}$$

Substituting for u^v in this last term its value from the differential equation

$$a_2^{iv} = 3\cdot\overline{14} + 2\cdot\overline{23} - 10\phi_3\overline{02} - 5\phi_4\overline{01}$$

Let $S_4 \equiv a_2^{iv} + 10\phi_3\overline{02} + 5\phi_4\overline{01}$. Then

$$S_4 = 3\cdot\overline{14} + 2\cdot\overline{23}$$

$$S_4' = 5\cdot\overline{24} + 3\cdot\overline{15} \qquad \text{or substituting}$$

$$= [illegible]\,\overline{24} - 3\cdot10\phi_3\overline{12} + [illegible]\,\phi_5\overline{01}$$

Let $S_5 \equiv S_4' - 3\phi_5 a_2 = 5\cdot\overline{24} - 30\phi_3\overline{12}$

$$S_5' = 5\cdot\overline{34} - 30[\phi_3'\overline{12} + \phi_3\overline{13}] + 5\cdot\overline{25}$$

$$= 5\cdot\overline{34} - 30[\phi_3'\overline{12} + \phi_3\overline{13}] + 5\{5\phi_4\overline{12} + \phi_5\overline{02}\}$$

Let $S_6 \equiv S_5' - 5\phi_5 a_2' = 5\cdot\overline{34} + (25\phi_4 - 30\phi_3')\overline{12} - 30\phi_3\overline{13}$

$$S_6' = (25\phi_4' - 30\phi_3'')\overline{12} + (25\phi_4 - 60\phi_3')\overline{13} - [illegible]$$

$$- 30\phi_3\{\overline{14} + \overline{23}\} + 5\,\overline{35}$$

$$S_6' = (25\varphi_4' - 30\varphi_3'')\overline{12} + (25\varphi_4 - 60\varphi_3')\overline{13} - 30\varphi_3(\overline{14} + \overline{23})$$

$$+ 5\{10\varphi_3\,\overline{23} + 5\varphi_4\,\overline{13} + \varphi_5\,\overline{03}\}$$

$$= (25\varphi_4' - 30\varphi_3'')\overline{12} + (50\varphi_4 - 60\varphi_3')\overline{13} - 30\varphi_3\left(\overline{14} + \frac{S_4 - 3\overline{14}}{2}\right)$$

$$+ 5\varphi_5(a_2'' - \overline{12}) + 25\varphi_3(S_4 - 3\overline{14})$$

Let $S_7 \equiv S_6' - 10\varphi_3 S_4 - 5\varphi_5 a_2'' = -(5\varphi_5 - 25\varphi_4' + 30\varphi_3'')\overline{12}$

$$+ (50\varphi_4 - 60\varphi_3')\overline{13} - 60\varphi_3\overline{14}$$

Let $X \equiv -(5\varphi_5 - 25\varphi_4' + 30\varphi_3'')$

$$Y \equiv (50\varphi_4 - 60\varphi_3')$$

$$Z = -60\varphi_3$$ Then

$$S_7 = X\overline{12} + Y\overline{13} + Z\overline{14} \tag{35}$$

$$S_7' = X'\overline{12} + (X + Y')\overline{13} + (Y + Z')\overline{14} + Y\,\overline{23} + Z(\overline{24} + \overline{15})$$

substituting for $\overline{23}$ and $\overline{15}$ their values

$$= \left(X' + \frac{Z}{15}\right)\overline{12} + \left(Z' - \frac{Y}{2}\right)\overline{14} + (X + Y')\overline{13} + \frac{Y}{2}S_4 + \frac{Z}{5}(S_4' + 2\varphi_3 - a_2)$$

Let $S_8 \equiv S_7' - \frac{Y}{2}S_4 - \frac{Z}{5}(S_4' + 2\varphi_3 - a_2)$

$$= \left(X' + \frac{Z^2}{15}\right)\overline{12} + (X + Y')\overline{13} + \left(Z' - \frac{Y}{2}\right)\overline{14} \tag{36}$$

$$S_8' = \left(X'' + \frac{Z'}{15}\right)\overline{12} + \left(2X' + Y'' + \frac{Z}{15}\right)\overline{13} + \left(X + \frac{Y'}{2} + Z''\right)\overline{14}$$

$$+ (X + Y')\overline{23} + \left(Z' - \frac{Y}{2}\right)(\overline{24} + \overline{15})$$

$$= \left(X'' + 3\frac{ZZ'}{15} - \frac{Y}{30}\right)\overline{12} + \left(Y'' + 2X' + \frac{Z}{15}\right)\overline{13} + \left(Z'' - \frac{2Y' + X}{2}\right)\overline{14}$$

$$+ \frac{X + Y'}{2}S_4 + \frac{\left(Z' - \frac{Y}{2}\right)}{5}(S_4' + 2\varphi_3 - a_2)$$

Let $S_9 = S_8' - \frac{X + Y'}{2}S_4 - \frac{1}{5}\left(Z' + \frac{Y}{2}\right)(S_4' + 2\varphi_3 - a_2)$

$$S_9 = \left(X'' + 3\frac{ZZ'}{15} - \frac{YZ}{30}\right)\overline{12} + \left(Y' + 2X' + \frac{Z^2}{15}\right)\overline{13} + \left(Z' - \frac{Y' + X}{2}\right)\overline{14} \quad (37)$$

$$S_9' = \left(X''' + \frac{3Z'^2 + 4ZZ''}{15} - 3Y'Z + \frac{1}{5}[illegible] + 2X\right)\overline{12} + \left(Y'' + 3X'' + \frac{ZZ'}{3} - \frac{YZ}{30}\right)\overline{13}$$

$$+ \left(Z''' - [illegible] + X' - \frac{Z^2}{[illegible]}\right)\overline{14} + \left(Y'' + 2X' + \frac{Z^2}{15}\right)\overline{13}' + \frac{1}{5}\left(Z'' - \frac{Y' + X}{2}\right)\left(\overline{14}' + [illegible] a_2\right)$$

$$[illegible] = [illegible] - \left(Y' + 2X' + \frac{Z^2}{15}\right)\overline{13}' - \frac{1}{5}\left(Z' + Y' - \frac{X}{2}\right)\left(\overline{14}' + [illegible]\right)$$

$$S_{10} = \left(X''' + \frac{1}{5}Z'^2 + \frac{4}{15}ZZ'' - \frac{1}{10}Y'Z - \frac{1}{30}YZ' - \frac{2}{30}X\right)\overline{12}$$

$$+ \left(Y'' + [illegible]X + \frac{3}{3}Z' - \frac{1}{30}ZY\right)\overline{13} + \left(Z'' - \frac{1}{2}Y' - \frac{3}{2}X' - \frac{Z^2}{30}\right)\overline{14} \quad (38)$$

Now we have four equations (35), (36), (37), (38) by which $\overline{12}$, $\overline{13}$ and $\overline{14}$ may be eliminated leaving

$$\begin{vmatrix} S_7, & X, & Y, & Z \\ S_8, & X' + \frac{Z^2}{15}, & Y' + X, & Z' - \frac{1}{2}Y \\ S_9, & X'' + \frac{ZZ'}{5} - \frac{YZ}{30}, & Y'' + 2X' + \frac{Z^2}{15}, & Z'' - Y' - \frac{1}{2}X \\ S_{10}, & X''' + \frac{4}{5}ZZ'' - \frac{3Y'}{10} - [illegible] + \frac{YZ' + 2X}{30}, & Y''' + 3X' + \frac{ZZ'}{3} - \frac{ZY}{30}, & Z''' - [illegible]Y - 3X' - [illegible] \end{vmatrix} = 0 \quad (39)$$

This is an equation in a_2 and its derivatives and functions derived from the coefficients of the original equation. It is of the tenth order and linear. It is the first associate of (34a).

To obtain the second associate, let

w represent any of the second associate variables. The diagonal of the determinant w has indices 0.1.2 Then we may write

$$w = \overline{012}$$

$$w' = \overline{013}$$

$$w'' = \overline{014} + \overline{023}$$

$$w''' = 2\cdot\overline{024} + \overline{123} - 10\varphi_3\,\overline{012}$$

Let $T_3 = w''' + 10\varphi_3 w = 2\cdot\overline{024} + \overline{123}$

$$T_3' = 3\,\overline{124} + 2\cdot\overline{034} + 2\,\overline{025}$$

$$= 3\cdot\overline{124} + 2\cdot\overline{034} + 2\cdot 5\varphi_4\,\overline{012}$$

Let $T_4 = T_3' - 10\varphi_4 w = 3\,\overline{124} + 2\,\overline{034}$

$$T_4' = 5\cdot\overline{134} + 3\cdot\overline{125} + 2\,\overline{035}$$

$$= 5\cdot\overline{134} + 3(\varphi_5\,\overline{012}) - 2(10\varphi_3\,\overline{032} + 5\varphi_4\,\overline{031})$$

Let $T_5 = T_4' + 3\varphi_5 w + 10\varphi_4 w' = 5\cdot\overline{134} + 20\varphi_3\,\overline{023}$

$$T_5' = 5\cdot\overline{234} + 60\varphi_3\,\overline{123} + 20\varphi_3'\,\overline{023} + 5\varphi_5 w + 10\varphi_3 T_3$$

$$T_6 = T_5' + 5\varphi_5 w - 10\varphi_3 T_3 = 5\cdot\overline{234} + 60\varphi_3\,\overline{123} + 20\varphi_3'\,\overline{023}$$

$$T_6' = (2\cdot 5\varphi_4 - 70\varphi_3')2\cdot\overline{024} + (20\varphi_3' - 5\varphi_5)\overline{023} + 60\varphi_3\,\overline{124}$$
$$+ T_3(80\varphi_3' - 25\varphi_4)$$

$$T_7 = T_6' - T_3(80\varphi_3' - 25\varphi_4) = (50\varphi_4 - 140\varphi_3')\overline{024} + (20\varphi_3' - 5\varphi_5)\overline{023} + 60\varphi_3\,\overline{124}$$

Proceeding in this way four equations

are obtained from which $0Z_4$ $0Z_3$, and TZ_4 can be eliminated. The result is

$$
\begin{vmatrix}
T_7, & X_1 & , Y_1 & , Z_1 \\
T_8, & X_1' + \frac{Z_1^2}{15} & , Y_1' + X_1 & , Z_1' - \frac{1}{2} Y_1 \\
T_9, & X_1'' + \frac{Z_1 Z_1'}{5} - \frac{Y_1 Z_1}{30} & , Y_1'' + 2X_1' + \frac{Z_1^2}{\;} & , Z_1'' \; Y_1' \; \frac{1}{2} X_1 \\
T_{10}, & X_1''' + \frac{4 Z_1 Z_1''}{5} - \frac{Z_1 Y_1'}{10} - 6\frac{Z_1'^2 + Y_1 Z_1' + X_1}{30} & , Y_1''' + 3X_1'' + \frac{Z_1 Z_1'}{3} - \frac{Z_1 Y_1}{30} & , Z_1''' \; Y_1'' - 2X_1' - \frac{Z_1^2}{30}
\end{vmatrix} = 0 \qquad (40)
$$

where $X_1 = 5\psi_5 = 20\psi_3''$ $\quad Y_1 = 50\psi_4 - 140\psi_3'$, $Z_1 = 60\psi_3$

This is also of the tenth order and linear.

The third Associate is the adjoint equation. Its form is *

$$v^{v} - 10\psi_3 v''' + (5\psi_4 - 20\psi_3')v' - (\psi_5 - 5\psi_4' + 10\psi_3'')v = 0 \quad 40$$

The first associate of the adjoint equation would be obtained from (39) by writing in it $-\psi_3$ for ψ_3

$5\psi_4 - 20\psi_3'$ for $5\psi_4$

$- - \quad -(\psi_5 - 5\psi_4' + 10\psi_3'')$ for ψ_5.

Now a little examination will show that this set of transformations among the coefficients which changes the differen—

tial equation into the adjoint equation and the adjoint into the differential equation. also transforms the first associate equation into the second, and the second associate into the first, and in particular S_7, S_8, S_9, S_{10}, X, Y, and Z respectively into T_7, T_8, T_9, T_{10}, X_1, Y_1, and Z_1, and vice versa.

Thus for the Quintic, at least, we have the result that:

the rth Associate of an equation equals the pth Associate of the adjoint equation when $r+p=3$. (42)

Before extending this theorem to any equation, in order to prepare the way, it will be considered in a different manner.

Let A_1 represent the differential equation of order n, and $u_1, u_2, u_3, u_4 \ldots u_n$ its fundamental solutions, A_2 the first associate and $a_2 \equiv \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$ any one of its solutions A_3 the second associate and a_3

any one of its solutions. and so on—

A_{n-1} is the adjoint equation and a_{n-1} any one of its solutions. A determinant will be represented by its diagonal in parenthesis. thus $a_2 \equiv (u_1 u_2')$ $a_3 \equiv (u_\kappa u_\lambda' u_\mu'')$ etc and in most cases the u's will be omitted.

Then $a_2 = (1\,2')$ $a_3 \equiv (1\,2'\,3'') = (\kappa\,\lambda'\,\mu'')$, writing more [illegible] κ, λ and μ being any three of the n suffixes, then $a_{n-1} = (1\,2'\,3''\,4'''\,{}^{iv} \ldots\, n{-}2^{(n-3)}\; n{-}1^{(n-2)})$ [illegible] a_n is the non vanishing constant usually denoted by Δ.

[illegible] a_4 [illegible] represent the first associate variable of the third associate equation, and $a_r A_s$ the r-1st associate variable of the s-1st associate equation.

Then the first associate variable of the third associate equation is [illegible] which expanded is

$$(2\,3'\,4'')(1\,5'\,6''\,8'''\,9^{iv}) - (1\,3'\,4'')(2\,5'\,6''\,8'''\,9^{iv}) + (1\,2'\,4'')(3\,5'\,6''\,8'''\,9^{iv}) - (1\,2'\,3'')(4\,5'\,6''\,8'''\,9^{iv})$$

In the case where n = 5, 6, 8 and 9

would be replaced by some of the five suffixes

[illegible]

to a single term viz: if 6, 8, 9. became

2, 3, 4., , $(2\dot{3}\ddot{4})(1\dot{5}\ddot{2}\dddot{3}\,4^{iv}) = -(2\dot{3}\ddot{4})(1\dot{2}\ddot{3}\dddot{4}\,5^{iv}) = -a_n(2\dot{3}\ddot{4})$

Thus, as found before, the first associate variable of the adjoint equation is a constant multiple of the second associate variable of A,

When n = six, a second associate variable of the adjoint equation, or $a_3 A_5$, is

$$\begin{vmatrix} (1\dot{2}\ddot{3}\dddot{4}\,5^{iv}), & (1\dot{2}\ddot{3}\dddot{4}\,6^{iv}), & (1\dot{2}\ddot{3}\dddot{5}\,6^{iv}) \\ (1\dot{2}\ddot{3}\dddot{4}\,5^{v}), & (1\dot{2}\ddot{3}\dddot{4}\,6^{v}), & (1\dot{2}\ddot{3}\dddot{5}\,6^{v}) \\ (1\dot{2}\ddot{3}\,4^{iv}\,5^{v}), & (1\dot{2}\ddot{3}\,4^{iv}\,6^{v}), & (1\dot{2}\ddot{3}\,5^{iv}\,6^{v}) \end{vmatrix} \quad a_3 A_5 ;$$

and this equals $a_n^2(1\dot{2}\ddot{3})$ or $a_3 A_1 \Delta^2$

Thus the second associate [illegible] variable of the adjoint equation, is a constant multiple of the first associate of A_1. This is expressed generally by $a_K A_{n-1} = a_\lambda A_1$, for all values of K and λ that make $\lambda + K = n$.

[illegible] this generally. [illegible]

will prove it for the last example.

Let the index of a term be the sum of the indices of its factors, and the dimension be the number of u's that are multiplied together in it. Then $(1\,3'\,6'')(4\,5'''\,7'''')$

[illegible]

a_n has dimension n and index $\frac{n-1 \cdot n}{2}$.

Any simple determinant of higher index than a_n must $= 0$ and also if it have higher dimension than a_n.

In the last example $n = 6$, the index is $3 \cdot \frac{5 \cdot 4}{2} + \frac{3 \cdot 2}{2} = 33$, and the dimension of every term is $3 \cdot 5 = 15$. Then

$a_3 A_5$ must equal $(1\,2'\,3''\,4'''\,5''''\,6)(2'\,3''\,4'''\,5''''\,6)(\kappa\lambda\mu)$

for this is of dimension 15 and index 33; and as each term contains any one suffix the same number of times every term must contain 1, 2, and 3 [illegible] times and 4, 5 and 6 twice. Then $(\kappa\lambda\mu) = (1\,2'\,3'')$. When we put $u_1 = 0$ both num-

bers of the equation contain zero [illegible] three times. The same is true when $u_1 =$ u_2 or u_3. and the contain zero as a factor twice when u_1 is made equal to either 4, 5, or 6. And considering each of the other pos[i]tions of the numbers [illegible] in identically the same way for twenty one [illegible] the equation $a_3 A_5 = a_{21}^2 (1\dot{2}\ddot{3})$ must be an identity. It should be noted that any change in one [illegible] of one member changes the other in the same manner.

In the same way it may be shown that $a_k A_{n-1} \equiv a_{n-k} A_1 . C$ where C is a constant. (44)

$$\left| \begin{matrix} (2\,\dot{3}\,\ddot{4}\,\dddot{5} \cdots \overset{n-2}{n}) & & & \\ & (1\,\dot{3}\,\ddot{4}\,\ddot{5} \cdots \overset{(n-3)}{n-1},\overset{(n-1)}{n}) & & \\ & & (1\,\dot{2}\,\ddot{4}\,\dddot{5}\,6 \cdots \overset{(n-2)}{n-1},\overset{(n-1)}{n}) + \left\{ (1\,2\,4\,5 \cdots \overset{(n-3)}{n-1},\overset{(n)}{n}) = X \right\} & \\ & & (1\,\dot{2}\,3\,\dddot{5}\,6 \cdots \overset{n-2}{n-2}\,\overset{n-1}{n}) + Y & \\ & & & \ddots \\ & & & (1\,2\,3\,4 \cdots n-2\; n) \end{matrix} \right| a_k A_{n-1}$$

This is a determinant of k rows and

columns, each constituent of which is a determinant with $n-1$ rows and columns. X and Y are elements in which the index (n) enters, and as $u^{(n)}$ may be replaced by the terms of the differential equation A_1 of order lower than $u^{(n)}$ columns will thus be introduced equal to those already in the determinant and consequently X and $Y = 0$.

If we consider the equation

$$a_k \cdot A_{n-1} = \left(1\,\dot{2}\,\ddot{3}\,\dddot{4}\,\overset{iv}{5}\cdots\overset{(n-1)}{n}\right)^{k-1}\left(k+1 . \overline{k+2}' \cdots \overset{(n-k-1)}{n}\right) = a_n^{k-1}\left(\overline{k+1} . \overline{k+2} \cdots \overset{(n-k-1)}{n}\right)$$

the left member is so arranged that the first n suffixes occur in $k-1$ elements while the rest of the suffixes occur in n elements. So also in the right member the suffixes occur in the same way. Then the dimension of each member is $(n-1)k = n(k-1)+n-k$, and the index of the left is $\frac{1}{2}\left\{k(n-1)(n-2)+k(k-1)\right\} = \frac{1}{2}\left\{n^2k-3nk-n^2+n\right.$

$$\left. +n^2-2nk-n+2k+n^2 k\right\} = \tfrac{1}{2}\,\overline{k-1}\,(n)(n-1)+\tfrac{1}{2}(n-k)(n-k-1)$$

[illegible] index [illegible],

[illegible]

little consideration will show that the equa-

tion must be an identity. Then it f[illegible]

that for all values of K. the Kth associate variable of an equation is a constant multiple of the $n-K$th. associate variable of the adjoint [illegible]tion.

When A_1 is self adjoint. $A_{n-1} = A_1$ and the theorem becomes. $a_K A_1 = a_{n-K} A_1$

All equations of Complimentary rank, associate to a self-adjoint equation are equal. (46)

The associate equations A_K and A_{n-K} are [illegible] to be of complimentary rank.

This completes the [illegible] relative to adjoint equations.

The question arises, is it true for any associate equations, that is, if A_r and A_{n-r} are associate equations of order m. then does.

viz $A_r = A_{m-2} A_{n-r}$ Turning to equation (39) and (40), When $\varphi_3 = 0$ and $\varphi_5 = 5\varphi_4'$. (39) reduces [illegible] equation [illegible] ninth order. [illegible] a linear relation between the

[illegible]

[illegible] a non-vanishing [illegible]

[illegible] a solution of [illegible]

Therefore eq. 42 does not [illegible]

[illegible] conditions generally.

Section IV

Conditions for the Self-Adjointness of Differential Equations

Any equation is self-adjoint [illegible] its invariants with odd suffix vanish.

Let n be the order of the equation.

The relations which exist between the [illegible] are

$$(-1)^n P_n = P_n - n P_{n-1}' + \binom{n}{2} P_{n-2}'' - \binom{n}{3} P_{n-3}''' + \binom{n}{4} P_{n-4}'''' \text{ etc. etc.} \qquad (17a)$$

These relations follow from those given by Mr. Craig in his treatise pp 490–493. For example note the results (?) pp 491 [illegible] pp 492. [illegible] that the equation be self-adjoint

$$P_2 = P_2$$

$$-P_3 = P_3 - 2P_2'$$

$$P_4 = P_4 - 3P_3' + 6P_2''$$

$$-P_5 = [illegible]$$

$$P_6 = P_6 - P_5' + P_4'' - P_3''' + P_2^{iv}$$ or generally

$$(-1)^\kappa P_{n-\kappa} = \sum_{\nu=0}^{\nu=n-\kappa} (-1)^\nu \binom{\nu+\kappa}{\kappa} P^{(\nu)}_{n-\kappa-\nu}$$

Had the equation been written with binomial coefficients this would become

$$(-1)^\kappa \binom{n}{\kappa} P_{n-\kappa} = \sum_{\nu=0}^{\nu=n-\kappa} (-1)^\nu \binom{\nu+\kappa}{\kappa}\binom{n}{\kappa+\nu} P^{(\nu)}_{n-\kappa-\nu}.$$

If we call $n-\kappa$ m and divide this by $\binom{n}{\kappa}$ it becomes

$$(-1)^\kappa P_m = P_m - m P_{m-1}' + \text{etc}$$

$$= \sum_{\nu=0}^{\nu=n-\kappa} (-1)^\nu \binom{m}{\nu} P^{(\nu)}_{m-\nu}$$

First let n be odd. Then

$$C = 2P_n - nP'_{n-1} + \binom{n}{2}P''_{n-2} - \binom{n}{3}P'''_{n-3} + \binom{n}{4}P^{iv}_{n-4} + - - + \text{ etc} \qquad (48)$$

$$2\theta_n = 2P_n - nP'_{n-1} + \binom{n}{2}\frac{n-3}{2n-3}P''_{n-2} - \binom{n}{3}\frac{(n-2)(2n-5)}{(n-4)(2s-3)}P'''_{n-3} + - +$$

$$\binom{n}{2}\frac{n-1}{2n-3}\theta''_{n-2} = \binom{n}{2}\frac{n-1}{2n-3}\left[P''_{n-2} - \frac{n-2}{2}P'''_{n-3} + 2\binom{n-2}{2}\binom{n-4}{1}\left(\frac{1}{2n-7}\right)P^{iv}_{n-4} + - + -\right]$$

Here it is seen that $(48) - 2\theta_n$ contains neither P_n nor P'_{n-1}, and that $(48) - 2\theta_n - \binom{n}{2}\frac{n-1}{2n-3}\theta''_{n-2}$ contains neither the first nor the second pair of terms, and from $(48) - 2\theta_n - \binom{n}{2}\frac{n-1}{2n-3}\theta''_{n-2} - \left(\binom{n}{4}\frac{n-1}{3}\frac{3}{2n-5}\right)\theta^{iv}_{n-4}$ [illegible] appear. Thus by subtracting certain multiples of the invariants and their derivatives from [illegible] disappear in pairs. The multiplier of $\theta^{(2\sigma)}_{n-2\sigma}$ will be

$$2\binom{n}{2\sigma}\frac{(n-1)}{(2\sigma)(n-2\sigma-1)} \equiv 2M_\sigma \text{ say}$$

Bearing in mind the conclusions of sections I and II. especially (22) and (23).

the coefficient of $P^{(2\kappa)}_{n-2\kappa}$ in (48) is $\binom{n}{2\kappa}$

" " " " " $2M_0\theta_n$ " " $M_0\binom{n}{2\kappa}\frac{n-2}{2\kappa-1}\frac{2\kappa-1}{2n-3} = M_0 C_0$ "

" " " " " $2M_1\theta''_{n-2}$ " " $M_1\binom{n-2}{n-2\kappa}\frac{n-4}{2\kappa-3}\frac{2\kappa-3}{2n-7} = M_1 C_1$ "

" .. .. - - - -

The coefficient of $D^{(2\kappa)}_{n-2\kappa}$ in $M_\sigma D^{(2\sigma)}_{n-2\sigma}$ is $M_\sigma \binom{n-2\sigma}{n-2\kappa}\binom{n-2\sigma-2}{2\kappa-2\sigma-1}\binom{2n-2\sigma-1}{2n-4\sigma-3} = M_\sigma C_\sigma$

It will [illegible] be shown that

$$\binom{n}{2\kappa} = \sum_{\sigma=0}^{\sigma=\kappa} M_\sigma C_\sigma \quad \text{or} \quad 1 = \sum \frac{M_\sigma C_\sigma}{\binom{n}{2\kappa}}$$

Let $\frac{M_\sigma C_\sigma}{\binom{n}{2\kappa}}$ be denoted by $m_\sigma c_\sigma$

then $m_0 c_0 = \frac{n-1!\; 2n-2\kappa-2!\; 2n-1 . 2\kappa!}{[illegible]\; n-2\kappa-1 . [illegible] n-1!}$.

$m_1 c_1 = \frac{[illegible]\; 2n-5 . 2\kappa!}{[illegible]\; 2n-3 . 2\kappa-2!\; [illegible]}$

. . . .

. . . .

. . . .

$m_\kappa c_\kappa = \frac{[illegible]}{2\kappa!\; 0!\; n-2\kappa-1!\; 2n-2\kappa-1!}$.

The general term is [illegible] form

$$m_\sigma c_\sigma = \binom{2\kappa}{2\sigma}\frac{n-1 . n-2 . n-3 \ldots n-2\kappa . 2n-4\sigma-1}{2n-2\sigma-1 . 2n-2\sigma-2 . 2n-2\sigma-3 \ldots 2n-2\sigma-2\kappa-1}$$

When $n=1$ $m_\sigma c_\sigma$ has a zero factor in the numerator for all values of σ except 0, therefore the series reduces to the term $m_0 c_0$ which contains a zero factor also in the denominator and becomes $\frac{-2\kappa-1!}{-2\kappa-1!} = 1$

For $n=2$ the numerator has a factor $n-2$ and the denominator $2n-4$ when $\sigma=0$ and 1, and so the series becomes $\frac{-2\kappa-1!\; 3}{3 . 2 . 1\; 2\kappa-3!} + \frac{2\kappa-2!\; 2\kappa . 2\kappa-1}{2\kappa-1!\; 2}$

Similarly for $n = 3, 4, 5 \cdots$

If $n = 2K$ the series is m_K [illegible]

$n = 2K-1$ it becomes $-m_K c_K + m_{K-1} c_{K-1}$

$$= \frac{2K-2!\,2K!}{2K-2!\,2K-1!\,2} - \frac{2K-2!}{2K-3!\,2} = 1$$

For $n = K-1$

$$m_0 c_0 = \alpha\, \frac{2K!\,2K-3}{2K-3!\,3!}(-1)^K \qquad \text{where } \alpha = \frac{K-2!\,K+1!}{2K!}$$

$$m_1 c_1 = \alpha.\, \frac{2K!\,2K-7\,.\,2K!}{2!\,2K-2!\,2K-5!\,5!}(-1)^K$$

$$m_2 c_2 = \alpha\, \frac{2K!\,2K!\,2K-11}{4!\,2K-4!\,7!\,2K-7!}(-1)^K$$

$\cdots$

$\cdots$

$$m_{K-3} c_{K-3} = \alpha\, \frac{2K!\,2K!\,2K-9}{6!\,2K-6!\,3!\,2K-3!}(-1)^{K-1}$$

$$m_{K-2} c_{K-2} = \alpha.\, \frac{\cancel{2K!\,2K!\,2K-5}}{4!\,2K-4!\,1!\,2K-1!}(-1)^{K-1}$$

$m_{K-1} c_{K-1} = 0 = m_K c_K$. Then

$m_0 c_0 - m_{K-2} c_{K-2} + m_1 c_1 - m_{K-3} c_{K-3} + m_2 c_2 - \text{etc}$

forms a series which is equal to unity

This is seen by taking the coefficient of y^{2K+3} from each member of the equation in which $(1-y)^{2K} \frac{d}{dy}(1+y)^{2K}$ is written equal to its expansion.

[illegible] $+ \binom{2K}{4} [illegible]$

$- [illegible] = 2K \frac{[illegible]}{6} [illegible]$

The coef. of y^{2K+3} in the product of the right members is $\Big\{ 4\binom{2K}{4} - 5\binom{2K}{5}\binom{2K}{1} + 6\binom{2K}{6}\binom{2K}{2} - 7\binom{2K}{7}\binom{2K}{3} + 8\binom{2K}{8}\binom{2K}{4} - + +$

$+ 2K\binom{2K}{4} - (2K-1)\binom{2K}{1}\binom{2K}{5} + (2K-2)\binom{2K}{2}\binom{2K}{6} - (2K-3)\binom{2K}{3}\binom{2K}{7} +$

and adding terms in the upper line to those below them this becomes $\binom{2K}{3}(2K-3) - \binom{2K}{4}\binom{2K}{1}(2K-5) + \binom{2K}{5}\binom{2K}{2}(2K-7)$

$- \binom{2K}{6}\binom{2K}{3}(2K-9) + \text{etc}$ which is the series $\sum_{\sigma=0}^{\sigma=n} \frac{m_\sigma c_\sigma}{\alpha} (-1)^K$

[illegible]

$$(-1)^K 2K \binom{2K-1}{K+1} = (-1)^K \frac{2K!}{K+1!\,K-2!} = (-1)^K \frac{1}{\alpha}$$

Therefore $\sum_{\sigma=0}^{\sigma=n} m_\sigma c_\sigma = 1$ (49)

Proceeding in a similar manner for other values of n it will be found that a similar [illegible]. Then for all values of n the coefficient of $P^{(2K)}$

in $(48) = 2\theta_n + M_1 \theta''_{n-2} + M_2 \theta^{iv}_{n-4} + + \text{etc} + M_K \theta^{2K}_{n-2K}$ [illegible] (50)

The coefficient of $P^{(2K+1)}_{n-2K-1}$ in the [illegible] of terms is formed from that of $P^{(2K)}_{n-2K}$ by giving [illegible] the same values and changing $2K$ to $2K+1$ [illegible]. Therefore, this [illegible]

If in (47a) n be even, the [illegible] relation between the coefficients is expressed by

$$0 = P'_{n-1} - \frac{n-1}{2} P''_{n-2} + (-1)^{v-1} \frac{n-1!}{v!\,n-v!} P^{(v)}_{n-v} \qquad v = 3.4.5 \cdots n-2 \tag{50}$$

In a way similar to the case when n is odd it may be shown that (51) is equal to a linear function of the invari[illegible]ants [illegible], say

$$(51) = \theta_{n-1} + N_3 \theta''_{n-3} + N_5 \theta^{iv}_{n-5} + \cdots + N_{n-3} \theta^{(n-3)}_{3} \tag{52}$$

Now the invariants in (50) and (52) have odd suffixes. Therefore when the invariants with odd suffixes vanish, (48) equals zero; and also (51) equals zero and the condition for self-adjointness is satisfied.

Thus the proposition with which the section begins is demonstrated.

It may be noticed however, that an equation may be self-adjoint when its invariants of odd suffix do not vanish but satisfy the linear relation forming the

Biographical Sketch of the Author

George Frederic Metzler, the son of George Frederic and Nancy Ann (Shannon) Metzler, was born July 17, 1853, at Westbrook, County of Frontenac, Ontario, Canada. His early education was received at the public schools of [illegible] then at different high schools. His Collegiate Education was received at Albert College, Belleville, Ontario (now consolidated with Victoria College [illegible] College [illegible])

At Albert College he took the degree A. B. in [illegible]. He has taught one year in a public school, two years in a high school and was called to teach in Albert College in [illegible]. He [illegible] the Johns Hopkins University in [illegible], and [illegible] in his eyes, entered again in 1887. [illegible]